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Inviscid Model of Two-Dimensional Vortex Shedding by a Circular Cylinder

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A discrete vortex model based on potential flow and boundary-layer interaction, rediscretization of shear layers, and circulation dissipation is developed to determine the characteristics of an impulsively started flow about a circular cylinder. The evolution of the flow from the start to very large times, lift and drag forces, Strouhal number, oscillations of the separation and stagnation points, and the vortex-street characteristics are calculated and compared with experiments.

Nomenclature

b = longitudinal spacing of vortices
C_D = drag coefficient
C_L = lift coefficient
C_{pb} = base pressure coefficient
c = radius of the cylinder
D = diameter of the cylinder, $D = 2c$
f_v = vortex shedding frequency
h = transverse spacing of vortices
i = $\sqrt{-1}$
j = an index
m = distance to nascent vortex from cylinder
N = number of vortices on a sheet
n = an index
q = total velocity at a point
Re = Reynolds number, UD/ν
St = Strouhal number, $f_v D/U$
s = distance along a sheet
δs = point vortex spacing
t = time or Ut/c for $U=1$ and $c=1$
Δt = numerical step size
U = velocity of the ambient flow
u = x component of velocity
v = y component of velocity
w = complex velocity potential
z = complex variable
Γ = circulation
γ = circulation per unit length
θ = angle measured from $(-c,0)$
λ = dissipation parameter
ν = kinematic viscosity of fluid
ρ = density of fluid
ω = vorticity

Introduction

The purpose of this paper is to describe a vortex model and a series of numerical experiments aimed at understanding the interaction between the boundary layer and the time-dependent wake of a circular cylinder immersed in an impulsively started steady viscous flow.

The separated flow about a circular cylinder has attracted some attention because of its practical and fundamental importance. Much of what is known about the consequences of separation has come from laboratory experiments. It has not yet been possible to develop a numerical model with which experiments may be conducted to explain the observed or inferred relationships between various parameters and to guide and complement the laboratory experiments. The principal difficulties are as follows:

1) Separation points. They represent a mobile boundary between two regions of vastly different scales. This, is turn, leads to complex physical nonuniformities in relatively narrow regions which cannot be handled within the framework of the boundary-layer theory.[1] Finite difference and marker and cell techniques require in such regions very small grid sizes which usually lead to prohibitively large computational times. The discretization of the continuous process of vorticity generation by the vortices in the vicinity of a mobile or fixed singular point (discrete vortex model) strongly affects the existing nonuniformities and promotes earlier separation. Attempts to preserve the prevailing conditions, for instance by limiting the influence of the nascent vortices while satisfying a relatively simple separation criterion, lead to hydrodynamical inconsistencies and nondisposable parameters.

2) Reynolds number. Finite-difference schemes for bluff-body flows are limited to relatively small Reynolds numbers, whether the scale of flow is assumed to be governed by a constant viscosity or by a constant eddy viscosity. The large recirculation region of the flow is often comprised of turbulent vortices, even when the boundary layer is laminar. Thus, the distribution, turbulent diffusion and decay of vorticity, and the interaction between the wake and the boundary layers cannot be subjected to numerical simulation without recourse to some heuristic turbulence models and inspired foresight. The representation of the wake by clouds of point vortices or discretized spiraling sheets (see, for example, Clements[2] and Fink and Soh[3] for comprehensive reviews and references) is not immune to scaling problems. In fact, not a particular Reynolds number but only a particular flow regime may be specified, depending on the separation criteria used (e.g., Pohlhausen's[4] laminar or Stratford's[5] turbulent boundary-layer separation criteria).

3) Three dimensionality. Even a uniform flow about a stationary cylinder exhibits chordwise and spanwise variations. These three-dimensional effects may play a major role in the stretching of vortex filaments and in the redistribution of vorticity in all directions. The numerical models are not in a position to account for such complex effects. One may hope to assess the effects of three dimen-

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sionality by means of two-dimensional numerical experiments.

However disconcerting the limitations of the numerical schemes may be, the primitive state of the description of the near wake of a cylinder can be improved only if the numerical experiments are pressed alongside the measurements. It is with this realization that a relatively simple discrete vortex model has been formulated and applied to impulsively started flow about a stationary cylinder. The application of the method to steady flow about an elastically mounted cylinder undergoing synchronous oscillations is presented in another paper.[6]

Discrete Vortex Method

The representation of the continuously distributed vorticity in restricted subregions of flow by a number of discrete line vortices is known as the discrete vortex model. Since its introduction by Rosenhead,[7] it has been used by an increasing number of investigators with varying degrees of success.[2,3] All applications of the method exhibited unsatisfactory features (e.g., vortex excursions and sheet kinking) and required the use of a number of nondisposable parameters unrelated to the conservation equations for the fluid they are to represent.[8,9] The principal difficulties are as follows:

1) Groups of discrete vortices moving under their own influence inevitably tend to a random distribution.[10,11]

2) The plane vortex sheets suffer from the Kelvin-Helmholtz instability,[12] independent of the numerical instabilities.

3) Feeding of vorticity at discrete steps at singular points (fixed or mobile separation points) distorts the existing flowfield and promotes earlier separation.

4) The determination of the mutual interaction between the boundary layers and the wake requires extremely accurate velocity profiles in the vicinity of the separation points.

5) Inviscid flow theory does not permit destruction or dissipation of vorticity. Different ad hoc modifications must be incorporated into the model, in violation of the vortex theorems, in order to account for the circulation reduction brought about by the rear shear layers, cross-wake mass and circulation transport, and the viscous and turbulent diffusion.

6) Contrary to observations, numerical simulation of symmetically-started flows remains symmetrical. Thus, it is necessary to introduce into the flow deliberate disturbances of limited strength and duration that will grow with no noticeable memory effects and enable the flow to reach a stable state intrinsic to viscous flows.

7) Computer storage and time. Most of the computation time goes into the calculation of velocities. Thus, vortex spirals or clouds must be coalesced somewhat arbitrarily to limit computer time.

Some, but not all, of the foregoing difficulties may be overcome. Fink and Soh[3] have shown that the cause of the late-time randomness or the chaotic motion of vortices is a consequence of the representation of the growing vortex sheet by an identifiable set of discrete vortices. They have arrived at this conclusion by demonstrating that the complex conjugate velocity of a segmented vertex sheet includes a logarithmic term that is not accounted for in the discrete vortex representation of the sheet. The remarkable consequences of this conclusion are that: 1) if the equivalent vortex is not placed at the midpoint of its segment through rediscretization of the sheet at each time interval, then the logarithmic term does not vanish and the computational error increases depending on the problem, the number of vortices, and the total time of computation; 2) the vortices which initially bisect the segment they are to represent do not continue to do so at the succeeding time intervals; 3) the use of finite vortex cores, accumulation of vortices at the center of the spiral, or other techniques only delay or minimize the accumulation of the errors resulting from the logarithmic term in an amount related to the distance between the point vortex and the center of the segment; and 4) the growth of the computational error may be significantly reduced by placing each discrete vortex at the midpoint of its segment, i.e., by placing the vortex at $z_j = 0.5(z_{j-1/2} + z_{j+1/2})$ at each time interval. Only through such a procedure can one make the logarithmic term vanish, provided that the radius of curvature of the sheet is not very small.

The calculations are then carried out at each time by representing the vorticity density by an entirely new set of equidistant vortices whose strengths are adjusted to give a good representation of that density. Fink and Soh[3] applied the rediscretization method to a number of flows past plate-like bodies with fixed separation points, and limited their calculations to relatively small times.

The present work determines the position of the mobile separation points through boundary-layer calculations, the strength and position of the nascent vortices through the use of the no-slip condition, and the evolution of the wake through the use of the method of rediscretization. A concerted effort is made to strike a balance between simplicity and realism.

Representation of the Flowfield

Complex Potential

The complex velocity-potential function $w(z)$ that describes the uniform flow U, a doublet at the origin to simulate the cylinder, m real vortices in the wake, and m image vortices in the cylinder is given by:

$$w(z) = -U\left(z + \frac{c^2}{z}\right) + \frac{i}{2\pi}\sum_{i}^{m}\Gamma_n\left[\ln(z - z_n) - \ln\left(z - \frac{c^2}{\bar{z}_n}\right)\right] \tag{1}$$

in which Γ_n and z_n represent the strength and location of the nth vortex, respectively, and c is the radius of the cylinder; an overbar indicates a complex conjugate. There are no images at the center of the cylinder because the vortices have been shed from it and leave circulation opposite to their own on the body.

The dependent and independent variables are normalized as

$$w(z)' = w(z)Uc^2, \quad z' = z/c, \quad \Gamma' = \Gamma/Uc, \quad t' = Ut/c, \quad u' = u/U$$

and then the primes are dropped from the dimensionless variables. It follows from Eq. (1) that the complex velocity at any point z is given by

$$\frac{dw}{dz} = -u + iv = z^{-2} - 1 + \frac{i}{2\pi}\sum_{1}^{m}\left[\frac{\Gamma_n}{z - z_n} - \frac{\Gamma_n}{z - 1/\bar{z}_n}\right] \tag{2}$$

The force components, normalized by $\rho c U^2$, are given by[13]

$$C_D + iC_L = i\frac{\partial}{\partial t}\sum \Gamma_n (z_n - z_{ni}) \tag{3}$$

where z_{ni} is the position of the nth image vortex. The velocities of the real and image vortices are related by

$$u_{ni} + iv_{ni} = (-u_n + iv_n)/\bar{z}_n^2 \tag{4}$$

Boundary Layer and Separation

The determination of the vorticity flux, the mobile separation points, and, hence, the interaction between the wake and the outer flow require the evaluation of the boundary layer at each time step. For an impulsively started flow about a circular cylinder, separation first occurs at $\theta = 180$ deg and $t = 0.351$ (Ref. 4). Subsequently, the separation points move rapidly at first and then slowly toward their quasisteady positions. Among others, Schuh[14] used the integral form of the unsteady momentum equation together with the steady potential-flow velocity distribution (not modified to account for the effect of the growing separation region on the outer flow) and found that separation occurs at $\theta_s = 107.2$ deg at $t=\infty$. Pohlhausen's steady flow approximation, again based on the velocity distribution obtained from the potential theory, gave $\theta_s = 109.5$ deg. In the present study, Schuh's unsteady momentum equation has been solved numerically with the modification that the outer flow is time dependent. The separation points reached $\theta_s = \pm 109.5$ deg at $t=0.98$. Then the first nascent vortices were placed at $\theta = 109.5$ deg in a manner to be described later, and the subsequent positions of the separation points were calculated independently using Schuh's time-dependent formulation and Pohlhausen's quasisteady state approximation. The results are shown in Fig. 1. Evidently, both methods yield nearly identical results as the separation point moves from $\theta = 109.5$ deg to about 80 deg. Thus, the time-dependent terms in the momentum equation have negligible influence on the prediction of the outer flow and the separation points for t larger than about unity. Accordingly, in all boundary-layer calculations for $t>1$, Pohlhausen's steady-state approximation has been used. In this context, it should be noted that the maximum value of $\partial\theta_s/\partial t$ for a separation point oscillation about a mean position, say $\bar{\theta}_s = 78$ deg, is considerably smaller than that in the range $80<\theta<110$ deg. Thus, the mobility of the separation points does not invalidate the use of Pohlhausen's method. (In the present calculations, the maximum velocity of the separation points is about 0.2 deg/time step.)

Nascent Vortices and the Separation Condition

The rate at which vorticity is shed into the wake may be closely approximated by[15]

$$\frac{\partial\Gamma}{\partial t} = 0.5U_s^2 \tag{5}$$

where U_s is the velocity of the outer flow at $\theta=\theta_s$. Thus, the strength of a nascent vortex is given by

$$\Gamma_{nv} = 0.5U_s^2\Delta t \tag{6}$$

where Δt is the time increment ($\Delta t = 0.125$ in the present calculations). Each separation point is treated independently and no symmetry is assumed or imposed. The nascent vortices are introduced into the flow at

$$z_{nv}^j = (1+m^j)\exp[i(\pi-\theta_s^j)] \qquad j=1,2 \tag{7}$$

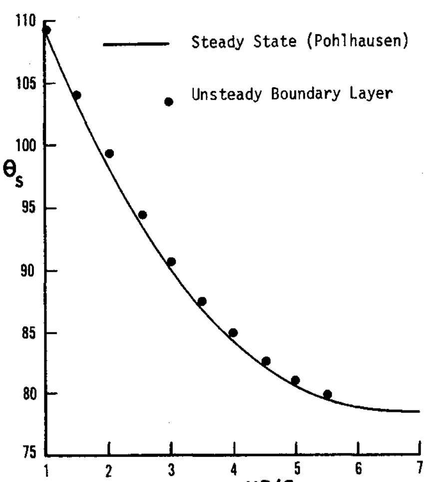

Fig. 1 Transient separation angle vs Ut/c.

with the requirement that the no-slip condition is satisfied at $z=\exp[i(\pi-\theta_s)]$. Thus, one has

$$1+m^j = \frac{1+|\Gamma_{nv}^j|/(2\pi U_s^j)}{1-|\Gamma_{nv}^j|/(2\pi U_s^j)} \tag{8}$$

in which m is the normal distance from the cylinder at $\theta=\theta_s$. The no-slip condition is satisfied exactly only at the instant of the introduction of each new vortex. It ceases to be valid during the remainder of the time step by the very nature of discretization, regardless of the method used to convect the nascent vortices. In the present calculations, the nascent vortices are convected with the velocity prevailing at z_{nv} at the time of their introduction.

Convection and Rediscretization of Vortex Sheets

The position of the vortices may be advanced in a number of ways,[16] some requiring iterations and all depending on the time interval used and the computation time available. A simple Eulerian scheme with a time step $\Delta t = 0.125$ was adopted; i.e.

$$z_n(t+\Delta t) = z_n(t) + q(t)\Delta t \tag{9}$$

where $q(t)$ is the complex velocity of the vortex to be convected. A large number of numerical experiments have been performed with a standard program by changing only the time step. Calculations with $\Delta t = 0.075$, 0.10, 0.125, 0.15, and 0.20 have shown that all the major parameters, such as lift, drag, and the position of the spiraling sheets, did not differ significantly. The frequent availability of 30-min time slots on a CDC-6600 computer for calculations up to $t=200$ dictated the choice of $\Delta t = 0.125$. The normalized circulation of the point vortices on a sheet took values from about 0.10 to 0.16. This was about 1-3% of the total circulation found in a vortex cluster.

All previous applications of the discrete vortex model to flow about a circular cylinder[17-20] used vortex clouds or an identifiable set of discrete vortices, affected only by coalescence, wall-proximity, and cancellation of oppositely signed vortices. This gave rise to a number of problems and remedies as previously discussed. The present model rediscretizes the vortex sheets at each time interval. To explain the method, let us consider a particular time t after the start of the motion and assume t to be sufficiently large so that there are three vortex sheets (two attached and one detached) and a number of coalesced vortices (Fig. 2). The vortices are assumed to be connected by straight-line segments of length $\delta s_n = |z_n - z_{n-1}|$, where $\delta s_n \simeq 0.1$ and the length of the sheet from z_1 (core of the spiral) to the nth vortex is calculated. Then the circulation per unit length, $\gamma_n(s)$, of the sheet at each z_n is determined from

$$\gamma_n(s) = 2\Gamma_n/(\delta s_{n+1} - \delta s_{n-1}) \tag{10}$$

and regarded as a piecewise continuous function. The use of small Δt and hence δs values resulted in a smooth $\gamma(s)$ variation along the sheet. Thus, no curve fitting was necessary.

The replacement of N vortices on a given sheet (attached or detached) by an equal number of equispaced point vortices is

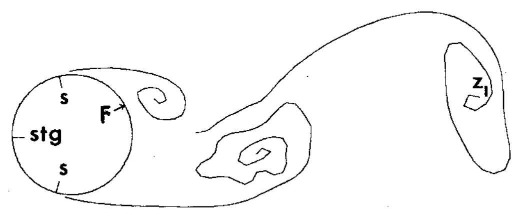

Fig. 2 Vortex sheets at an arbitrary time t.

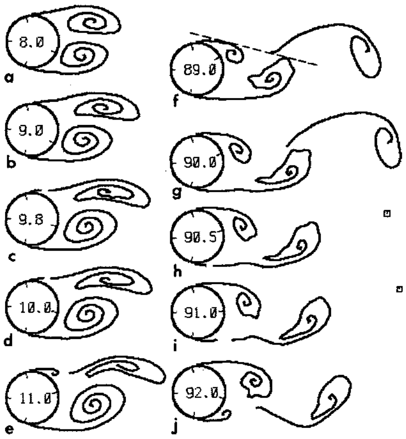

Fig. 3 Onset of asymmetry and shedding of vortices.

accomplished by starting at the core of the spiral, z_1 (assumed to remain unaffected by rediscretization), and placing point vortices at intervals of

$$\gamma s = \sum_{1}^{N} \delta s_i / (N-1) \tag{11}$$

measured along the straight-line segments connecting the original set of point vortices. Vortex strengths were assigned to the new vortices through the use of the distribution function $\gamma(s)$, such that the vorticity per unit length at a new position is the same as for the original sheet at that s. This is achieved simply by using a linear interpolation of $\gamma(s)$ values as

$$\gamma(s) = \gamma_{n-1} + (s - s_{n-1})(\gamma_n - \gamma_{n-1})/(s_n - s_{n-1}) \tag{12}$$

The rediscretization process does not change either the strength or the position of the first and the last vortex on a given spiral.

The final step in this procedure is to insure that there has been no net loss or gain of vorticity and no change in moment of vorticity due to linearized approximations in rediscretization. For this purpose, the total circulation in each sheet was calculated before and after the rediscretization. The difference, if any existed, was uniformly distributed among N-2 vortices. The moment of vorticity was then calculated and found that it remained constant within $\pm 1\%$. No corrections were made for the small differences for reasons that will become clear later in connection with the discussions of circulation reduction.

Treatment of Particular Effects

Inducement of Asymmetry

In order to induce the shedding of the first vortex and the formation of an asymmetric street, the developing flowfield must be perturbed. A number of different methods may be used, all amounting to a small change in position and/or strength of a few vortices on one side of the street over a short time interval. In the present study, numerous methods have been tried to find the time t at which the symmetric vortex spirals are most susceptible to small disturbances. It was found that the disturbances introduced at times $t<4$ should be considerably larger than those introduced at times $t>5$. The rate of vorticity passed through a minimum and the symmetric feeding sheets considerably weakened over the interval $5<t<9$. Accordingly, the point vortices on one side of the wake were given small lateral displacements over the said period with magnitudes gradually increasing with time.‡ The total displacement over the said interval was 0.16. It must be emphasized that there is less arbitrariness in the flow-perturbation method than it may appear otherwise. In fact, the late-time flow does not exhibit any memory effects to any reasonable initial disturbances. Space limitations prevent us from presenting a fuller account of the effects of disturbance time, type, and magnitude on the initial stages of the flow.

Shedding of Vortices.

As the vortices grow symmetrically, part of the shear layer between the nascent vortex and the vortex spiral (the one at top and displaced further downstream, Fig. 3) begins to develop instabilities (presumably Tollmien-Schlichting instabilities[4]) and is drawn across the wake in response to the base pressure reduced by the action of the vortex growing at the bottom. This nearly corresponds to a time when the sheet drawn in has least circulation. The stretching, diffusion, and dissipation of vorticity break up the deforming turbulent sheet and, thereby, the further supply of circulation to the detached sheet at the top. This corresponds to the shedding of the first vortex.

The growing vortex (bottom, Figs. 3f and 3g) still continues to grow (but at a decreasing rate) and entrains part of the oppositely signed vorticity left in the wake by the tail of the detached sheet and the irrotational fluid drawn from outside through the opening created by the shedding of the top vortex. The shedding process for the bottom vortex does not commence until the circulation in its feeding sheet decreases nearly to its minimum, making the sheet most susceptible to turbulent diffusion. Simultaneously, the sheet deforms, diffuses, and is drawn across the wake by the action of the base pressure and the vortex growing on the other side of the wake. Then the shedding cycle repeats itself. This mechanism, exhibited by the numerical experiments, is quite similar but not identical to that suggested by Gerrard.[21]

The separation point at the side of the cut sheet just passes through its minimum angle. The sheet at the opposite side of the wake bears nearly the maximum circulation relative to any other time, and its separation-point angle just passes through the maximum. The motion of the stagnation point is such that it is 180 deg out of phase with the separation points, i.e., the stagnation point is below the x axis when a vortex is shed from the bottom of the cylinder.

The numerical calculations cannot be carried out indefinitely without actually cutting the sheet that is drawn across the wake because of extreme demands placed on the rediscretization process (Fig. 4). It was necessary, therefore, to seek a relationship between the variables calculated which would signal a precise time for the cutting of the deformed sheet. Only after numerous numerical experiments with rediscretized and nonrediscretized versions of the same model (Fig. 5) was the importance of the cutting time relative to the time of occurrence of the minimum $\partial\Gamma/\partial t$ discovered. The mechanism of the cutting consisted of the removal of a single point vortex from the sheet in an interval $0.4<x<0.8$ during one time step only. The sheet was cut immediately after $\partial\Gamma/\partial t$ reached its minimum at a point where $0.4<x<0.8$. This corresponded to 59-61 deg phase difference between the lift extremal and the cutting time. Other cutting times invariably resulted in unequal rates of vorticity for the corresponding positions of the top and bottom separation points. Thus, the model has shown that a sheet is cut when and where it is weakest.

The foregoing was not in any way dependent on the initiation of asymmetry. One experimental accident, where the inducement of asymmetry was turned off, showed that the cutting mechanism (actuated by a slight fluctuation in $\partial\Gamma/\partial t$) is sufficient to induce and sustain asymmetry. In nature, the relative weakness of one of the two sheets feeding a pair of nearly symmetrical growing vortices, at a time when both are receiving reduced levels of vorticity, may very well be the cause of the events leading to the shedding of the first vortex.

The shedding of the vortices and the cutting process alternate with a time period equal to one-half the Strouhal

‡$\Delta x = 0.04[1 - \cos\{\pi(t-5)/4\}]$ for $5<t<9$.

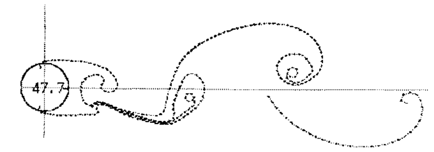

Fig. 4 Continuous rediscretization of vortex sheets.

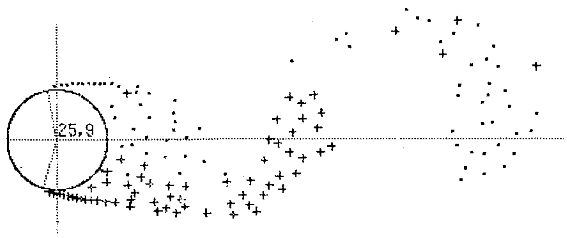

Fig. 5 Wake development without rediscretization.

period. When a vortex sheet is cut, say at the lower side of the cylinder, the sheet that was previously detached from the top is coalesced into a single vortex at its vorticity center of gravity. At the time of its coalescence, the core of the vortex spiral moves a distance of about seven radii downstream. The calculations have conclusively shown that the process of coalescence at this distance does not alter the continuity of any variable (e.g., lift, drag, vorticity flux).

Reduction of Circulation

All discrete vortex models of flow about bluff bodies[2,16-20] have shown that the concentrated vortices of the wake would contain about 80% of the shed vorticity in contrast to the experimentally measured value of around 60%. This feature of the model, which affects the position of the separation points, lift force, the formation length of the near wake, and, to a lesser extent, the Strouhal number and the relative vortex spacing in the stable and unstable regions of the street remained unresolved. The 20% reduction in circulation was realized through various means: 1) removal of the vortices from the calculation whenever they came nearer than a specified distance to the rear face of the body[16,20] or to their oppositely signed images; 2) removal of two oppositely signed real vortices when they came within a prescribed distance from each other; 3) use of rear shear layers[16,20] to generate oppositely signed vorticity to cancel that generated at the nearest primary shear layers; and 4) the reduction of the strength of the nascent vortices by a suitable percentage right at the point of their appearance.[17] Evidently, the use of a viscous core[19] has no direct bearing on the circulation reduction.

None of the methods cited above can, independently or in combination, account for either the magnitude or duration of the circulation reduction. Methods (1) and (2) produce a reduction of about 15-20%. Both are random and highly dependent on the magnitude of Δt. Method 1 changes the net circulation and may cause erratic changes in lift. Method 3 does not permit the formulation of a defensible mechanism whereby the rear boundary layer and its separation can be formulated without some sweeping assumptions. The proximity of the vortices and their images to the cylinder may cause erratic changes in the velocity distribution. Furthermore, it is hard to accept that the rear shear layers can bring about a 40-50% reduction in circulation. Method 4 simply is not acceptable. Finally, none of the methods accounts for the observed fact that the strength of the vortices continues to decrease with time or distance.[21,22]

The question in devising a model is not whether the circulation must be reduced, but rather how. The primary difficulty in devising a circulation-reduction mechanism lies in the assumption of an inviscid wake, which implies that the strength of the vortices must remain constant in time, i.e., $D\omega/Dt=0$.

In search of a suitable mechanism, numerous numerical experiments have been carried out with one or more of the methods just cited, with or without rediscretization. These experiments led to the following conclusions: 1) None of the the three methods can yield a circulation reduction in conformity with the observations. 2) The periodic entrainment of oppositely signed vorticity across the wake causes only about a 10% reduction in circulation. 3) If the wake of a separated viscous flow is to be modeled by discrete vortices, then the relationship $D\omega/Dt=0$ must be disregarded so that one can recognize and suitably account for the reduction of vorticity.

The transition to turbulence moves upstream in the shear layers as the Reynolds number is increased from about 10^3 to 5×10^4. At $Re=5\times10^4$, it reaches the shoulder of the cylinder.[23] It does not move appreciably further upstream before the critical Reynolds number is reached. The present numerical model, based on the laminar boundary-layer separation criterion, is most applicable in the upper range of the subcritical Reynolds numbers where the shear layers are turbulent for all values of $x>0$.

Turbulence evolves and diffuses into the vortex spiral. The width of the turbulent shear layer and the entrainment into it are expected to increase with the length of the shear layer. Thus, in devising a heuristic model for the dissipation of vorticity, it was required that the shape of the $\gamma(s)$ function and the center of vorticity remain the same before and after the reduction of vorticity at each time step of the calculation, and that the dissipation increase approximately with the length of the spiral. This requirement led to the hypothesis that every vortex in the wake loses its strength in an amount proportional to its current strength and position after the rediscretization of the sheet. The constant of proportionality λ was chosen as follows: λ increasing linearly from 0 to 0.1 for $t<5$; $\lambda=0.01$ for $t>5$ in the region $0<x<10$; and λ decreasing from 0.01 to 0 at large distances. The particular form of $\lambda(x)$ for $x>10$ had very little or no influence on the major flow parameters. Often a simple linear decrease to zero at about $x=20$ was sufficient. Other values of $\lambda(x,t)$ have been used in the course of calculations in order to determine the effect of the magnitude and shape of $\lambda(x,t)$ on all the parameters calculated. These will be discussed later. It is sufficient to note that this mechanism caused reduction in the formation region of about 20%.

There are two other processes whereby the circulation is reduced. The first is the reduction resulting from the proximity of the vortices to the cylinder. Whenever a point vortex came closer to the cylinder than a radial distance smaller than $\Delta r=0.04$, that point vortex is removed from the flowfield. The part of the sheet connected to the nascent vortex is excluded from this process for obvious reasons. The annihilation of circulation in this manner resulted only in a small reduction (about 10%) in the total circulation. This fact was the first indication of the need for a mechanism based on turbulence dissipation.

The second of the two mechanisms concerns the entrainment of the tail of the detached sheet into the vortices across the wake. As noted earlier, the deformed sheet is drawn into the region between two attached sheets because of the reduced base pressure. The circulation in the tail of the deformed sheet gradually ends up in the attached sheet across the wake. In the model, a similar procedure is used. That portion of the tail which has penetrated into the mixing region beyond a line roughly tangent to the two attached vortex spirals is removed from the flowfield and its circulation is uniformly distributed among the vortices along the sheet across the wake (see Fig. 3f). The position of the line severing the tail of the detached sheet is not of particular importance. One could have used other equally suitable procedures without affecting either the results or the magnitude of circulation loss.

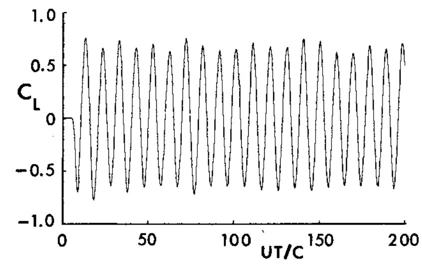

Fig. 6 Lift coefficient as a function of Ut/c.

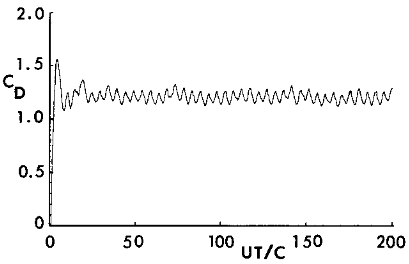

Fig. 7 Drag coefficient as a function of Ut/c.

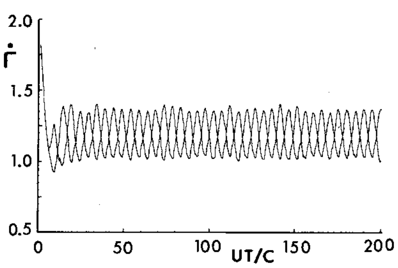

Fig. 8 Rate of change of circulation as a function of Ut/c.

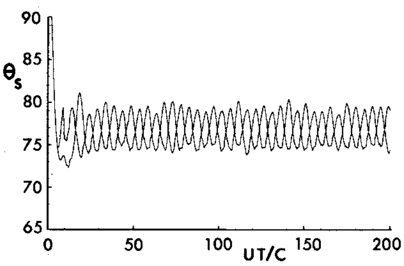

Fig. 9 Separation angles as a function of Ut/c.

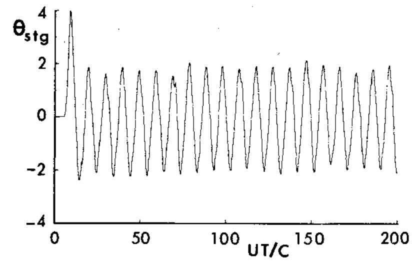

Fig. 10 Stagnation angle as a function of Ut/c.

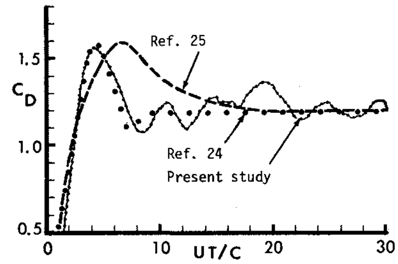

Fig. 11 Comparison of measured and calculated drag coefficients.

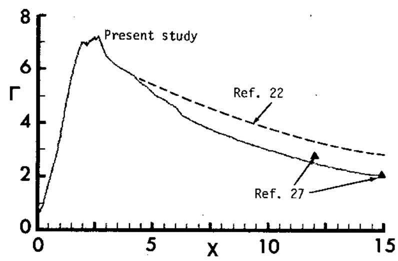

Fig. 12 Circulation of a single vortex as a function of distance.

Discussion of Results and Sensitivity Analysis

Figures 6-10 show the lift and drag coefficients, the rate of vorticity shedding, and the separation and stagnation angles as a function of time or relative displacement. The lift coefficient has a mean amplitude of about 0.65 and a frequency of $f_v = 0.1025$, corresponding to a Strouhal number of $St = 0.205$. The drag coefficient reaches a maximum at $t = 4.2$ and then rapidly decreases to its ultimate value of about 1.2. The fluctuations of C_d have a frequency twice the vortex shedding frequency and an amplitude of about 0.06. The overshoot of the drag coefficient is a direct consequence of the rapid accumulation of vorticity in the symmetrically growing vortices.

Figure 11 shows a comparison of the calculated and measured drag coefficients at relatively smaller times. Experiments at the early stages of the impulsive flow are very sensitive to the evolution of velocity. Consequently, even though the peak and steady-state values of C_D agree quite well, the time at which C_D reaches its peak value and the time interval during which C_D decreases to its steady-state value are not necessarily identical.[24,25]

The mean value of $\partial\Gamma/\partial t$ is about 1.16. This yields a base pressure coefficient $\bar{C}_{pb} = 1 - 2\,\overline{|\partial\Gamma/\partial t|} = -1.32$, which is somewhat larger than the experimental value[26] of -1.2 in the range $2 \times 10^4 < Re < 10^5$. No attempt has been made to increase λ, so as to increase the calculated base pressure.

The separation points oscillate nearly sinusoidally with an amplitude of $\Delta\theta = 3$ deg about a mean value of $\theta_s = 77$ deg. The stagnation point oscillates with an amplitude of $\Delta\theta = 2$ deg. Furthermore, the oscillations of the separation and stagnation points are 180 deg out of phase. These results are in good agreement with those obtained experimentally.[27] The calculated mean value of the separation angle is somewhat lower than the commonly accepted value of 82 deg, primarily due to the strong influence of the nascent vortices on the velocity distribution near the separation.

The longitudinal and transverse spacing of the vortices in the fully developed region of the street were found to be $b/D = 4.5$ and $h/D = 1.08$.

Figure 12 shows the circulation retained by a vortex sheet from its inception to its subsequent coalescence and convection into the far wake. Also shown in this figure are the two experimental points reported by Bloor and Gerrard,[28] and the mean curve drawn through the data obtained by Schmidt and Tillman.[22] The comparison of the results is better than expected, in view of the fact that the experimental determination of Γ through direct or indirect methods is an extremely difficult and approximate task. The percent circulation retained in a vortex at any x may be obtained by dividing the Γ values in Fig. 12 by $\Gamma_t = 11.3$, the total circulation shed in one period ($T = 9.75$ and $St = 0.205$). It is seen that at $x = 4$, 50%, $x = 8$, 32%, and at $x = 12$, 26% of the total circulation is retained in the vortices.

The foregoing does not imply that the particular variation assigned to the circulation loss is unique; it simply recognizes the need to account for the transition to turbulence in the free-shear layers, entrainment of fluid into these layers, and the diffusion and decay of vorticity. A series of calculations have been performed by changing the magnitude of λ while retaining its functional form. The results have shown that for $\lambda = 0.005$, $\bar{C}_D = 1.35$, $C_L = 0.90$, $St = 0.195$; for $\lambda = 0.01$, $\bar{C}_D = 1.2$, $C_L = 0.66$, $St = 0.205$; and for $\lambda = 0.015$, $C_D = 1.1$, $C_L = 0.5$, and $St = 0.22$. Evidently, $\bar{C}_D$ and St are least sensitive to the rate of dissipation. Furthermore, the stronger the vortices (smaller λ), the smaller is the Strouhal number. These results may explain in part the lack of scatter in C_D and St in laboratory experiments and the difficulty of obtaining consistent lift data.

Numerous calculations have been performed to determine the effect of changing the independent variables. The time interval was varied from 0.1 to 0.2, while keeping all else constant. C_D changed only 1%; C_L, 0.05%, and St remained constant at 0.205.

The shape of the dissipation function has been changed to $\lambda = \sigma / x$ for $x > 1$, where σ is a constant. This form has shifted the region of larger dissipation closer to the cylinder. The results obtained with $\sigma = 0.02$ have shown that C_L and C_D vary not more than 10% from those shown in Figs. 6 and 7. The separation point remained at almost exactly the same average value. The percent circulation retained by the concentrated vortices increased with a corresponding decrease in St (0.205-0.193), indicating that not only the vortices in the formation region, but also those further downstream, determine the transport velocity of the vortices and hence St. The experience gained from the numerical experiments has shown that λ is fairly constant in the near wake. It decreases gradually with distance beyond the formation region.

To determine further the sensitivity of the model to dissipation and the relationship between some of the most important parameters, an additional series of numerical experiments has been performed. Specifically, the separation angle was determined not in the manner described in the standard model, but rather by suitably extrapolating§ the velocity profiles beyond the point of maximum velocity. The use of such an extrapolation alleviated the effect of the nascent-vortex proximity on the separation point and resulted in larger separation angles ($\bar{\theta}_s = 81$ deg). When the shape of the dissipation function was kept exactly as in the standard run, the drag coefficient reached an average value of 1.3 and $C_L = 0.8$. The Strouhal number decreased to 0.19. Thus, it was apparent that the value of λ should be increased to about 0.013 on the basis of previous sensitivity analysis. Such an increase in λ resulted in C_D, C_L, and St values identical to those obtained with the standard run. The position and the amplitude of oscillation of the separation and stagnation points remained unchanged. This experiment has suggested that it is possible to match the observed and calculated separation points and resistance with one disposable parameter. No attempt has been made to find the precise value or shape of the dissipation function because of the primary concern wih the interrelationships between the various parameters. It is evident that the numerical modeling of a turbulent wake does require dissipation and that it can be quantified only by additional measurements that will complement the numerical experiments.

Conclusions

A discrete vortex model based on the rediscretization of the shear layers, boundary layer-wake interaction, and a heuristic model of circulation dissipation has been developed to determine the interaction between the major parameters of the flow past a circular cylinder. The numerical experiments generated values in reasonable agreement with all the quantities available for comparison with experiment. The insensitivity of the Strouhal number to changes in circulation reduction explained the success of the previous models[2,16] in predicting the Strouhal number fairly accurately. The reason for relatively smaller separation angles predicted by numerical models has been shown to be the sensitivity of the velocity profile (beyond the point of maximum velocity) to the process of discretization of vorticity near the point of separation. A method has been suggested to overcome this difficulty. Finally, the results further elucidated the mechanism of vortex shedding and the role played by the base pressure.

§Using $u/U_m = A(t)\theta + B(t)\theta^3 + C(t)\theta^5$ where A,B, and C are determined at each time step using the calculated velocities and requiring $u/U_m = 1$ and $\partial u/\partial\theta = 0$ at $\theta = \theta_m$ and $u = U_l$ at $\theta_l = \theta_m + 4$ deg.

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